BIFURCATION THEORY

X, Y Banach spaces. We want to study problems of the form

$$F(\underbrace{\lambda}_{\to\ \text{parameter}}, u) = 0, \quad F: \mathbb{R}\times X \to Y, \quad C^2$$

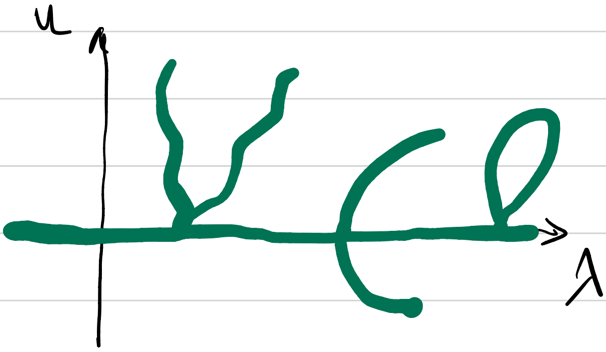
We shall consider situation in which

$$F(\lambda, 0) = 0 \quad \forall \lambda \in \mathbb{R} \rightsquigarrow u=0 \text{ is trivial sol}$$

Look for <u>non</u> trivial sol, define

$$S = \{ (\lambda, u) \in \mathbb{R}\times X : \ u \neq 0, \ F(\lambda, u) = 0 \}$$

We want to find some values of λ for which there are 1 or more solutions



The values of λ for which solutions branch off are called

BIFURCATION POINTS

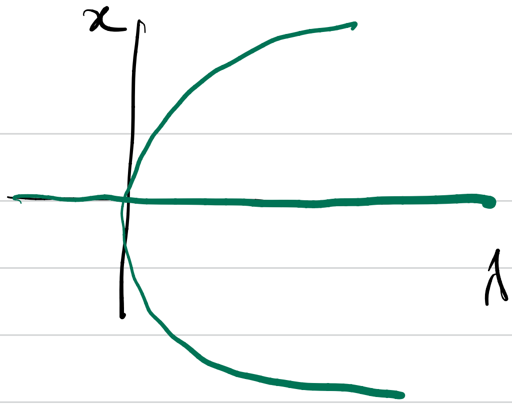
Def λ^* is a <u>bifurcation point</u> for F if $\exists\ (\lambda_n, u_n)_n \in \mathbb{R}\times X$ with

$$\begin{cases} u_n \neq 0 & \forall n \\ F(\lambda_n, u_n) = 0 & \forall n \\ (\lambda_n, u_n) \to (\lambda^*, 0) \end{cases}$$

<u>Trivial example</u> $x^3 - \lambda x = 0$, $F(\lambda, x) = x^3 - \lambda x = x(x^2 - \lambda)$

$F: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$. $x = 0$ is a sol $\forall \lambda$

If $\lambda < 0$ is the only sol
$\lambda > 0$ also $\pm\sqrt{\lambda}$ are sol



$\Rightarrow \lambda = 0$ is a bif. point

<u>Necessary condition</u> For λ^* to be a bifurcation point then $d_u F(\lambda^*, 0)$ not invertible!

otherwise if $d_u F(\lambda^*, 0)$ is invertible with bd inv, by IFT $\exists$ neigh $B^{\mathbb{R}}_{\varepsilon}(\lambda^*) \times B^X_{\delta}(0)$ st

$$F(\lambda, u) = 0 \quad \text{in} \quad \Leftrightarrow \quad u = 0$$

↳ in this neigh sol is unique and we know $u = 0$ is sol

<u>Interesting case</u> $F(\lambda, u) = \lambda u - G(u)$ with $G \in C^1$

Then if λ^* bif point $\Rightarrow \lambda^* \in \sigma(G'(0))$

Indeed: $\quad d_u F(\lambda^*, 0) = \lambda^* \mathbb{1} - G'(0)$

so $d_u F(\lambda^*, 0)$ is invertible $\Leftrightarrow \lambda^* \in \rho(G'(0))$
not " $\Leftrightarrow \lambda^* \in \sigma(G'(0))$

<u>Rem</u> it is not suff. : λ^* can be in $\sigma(G'(0))$ but not be bif. points,

$$F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2, \quad F\left(\lambda, \binom{x}{y}\right) = \lambda \binom{x}{y} - \underbrace{\begin{pmatrix} x + y^3 \\ y - x^3 \end{pmatrix}}_{G\binom{x}{y}}$$

then $G'\left(\binom{0}{0}\right) = \mathbb{1}$ so $\lambda^* = 1$ is eigenvalue.

But $F\left(\lambda, \begin{pmatrix} x \\ y \end{pmatrix}\right) = 0 \iff \lambda \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y^3 \\ y-x^3 \end{pmatrix}$

$\Rightarrow x^4 + y^4 = 0 \Rightarrow x = y = 0$ (no bifurcation)

↳ multiply 1^{st} eq by x^3, 2^{nd} by y^3 and sum

LYAPUNOV - SCHMIDT REDUCTION

It is a general method to deal with bifurcation problems:

$$F \in C^2(\mathbb{R} \times X, Y) \quad st \quad F(\lambda, 0) = 0 \quad \forall \lambda$$

We know that $d_u F(\lambda^*, 0)$ not invertible if λ^* bifurcation.

We assume that

($X = V + W$, $V \cap W = \{0\}$, V, W closed)

(A1) $V := \ker d_u F(\lambda^*, 0) \neq 0$ is complementable

(A2) $R := \operatorname{Im} d_u F(\lambda^*, 0)$ is closed and complementable
($Y = R + Z$, R, Z closed, $R \cap Z = \{0\}$)

Let $P: Y \to Z$ (proj over complement of range)
$Q: Y \to R$ (proj over range)

P, Q are continuous because R, Z are complementable

We project $F(\lambda, u) = 0$ using P, Q and we also decompose

$$u = v + w \in V + W \quad \text{(kernel + compl)}$$

$$F(\lambda, u) = 0 \iff \begin{cases} P\,F(\lambda, v+w) = 0 \\ Q\,F(\lambda, v+w) = 0 \end{cases}$$

GOAL: solve the Q eq:

start by writing

$$F(\lambda,u) = \underbrace{F(\lambda_0)}_{=0} + \underbrace{d_u F(\lambda_0)[u]}_{= L_\lambda} + \underbrace{f(\lambda,u)}_{= F(\lambda u) - L_\lambda u}$$

Apply Q, $u = v + w$:

$$0 = Q F(\lambda, v+w) = Q\big(L_\lambda (v+w) + f(\lambda, v+w)\big)$$

Set $\phi(\lambda, v, w) := Q L_\lambda (v+w) + Q f(\lambda, v+w)$

We know $\phi \in C^2(\mathbb{R} \times V \times W, R)$

$$\phi(\lambda^*, 0, 0) = Q f(\lambda^*, 0, 0) = Q F(\lambda^*, 0) = 0$$

Goal: given (λ, v), find $w = w(\lambda, v)$ solving $\phi(\lambda, v, w) = 0$

so $d_w \phi(\lambda^*, 0, 0) : W \to R$ must be invertible

$$d_w \phi(\lambda^*, 0, 0)[\hat{w}] = Q L_{\lambda^*} \hat{w} + Q\, d_u f(\lambda^*, 0)[\hat{w}]$$

Q proj over $R = \operatorname{Im} \underbrace{d_u F(\lambda^*, 0)}_{L_{\lambda^*}}$

$$= L_{\lambda^*} \hat{w} + Q\Big(\underbrace{d_u F(\lambda^*, 0)[\hat{w}]}_{= L_{\lambda^*}\hat{w}} - L_{\lambda^*} \hat{w} \Big)$$

$$= L_{\lambda^*} \hat{w}$$

Consider: $L_{\lambda^*} : \underset{\text{complement of } \ker L}{W} \to \underset{\text{range } L_{\lambda^*}}{R}$

- continuous ✓
- surjective ✓
- injective ✓
- W, R closed $\rightsquigarrow$ Banach

open mapping th.

$\rightsquigarrow L_{\lambda^*}|_W : W \to \mathbb{R}$ is invertible with bd inverse!

We apply IFT to Φ and solve it locally with respect to W.
Precisely we find a function

$$\gamma : B_\varepsilon^{\mathbb{R}}(\lambda^*) \times B_\varepsilon^V(0) \longrightarrow B_\delta^W(0)$$
$$\lambda \quad , \quad v \longrightarrow \gamma(\lambda, v)$$

so that $\Phi(\lambda, v, \gamma(\lambda, v)) = 0 \quad \forall (\lambda, v) \in B_\varepsilon^{\mathbb{R}}(\lambda^*) \times B_\varepsilon^V(0)$

and it is the unique sol in $B_\varepsilon^{\mathbb{R}}(\lambda^*) \times B_\varepsilon^V(0) \times B_\delta^W(0)$

$$\gamma(\lambda, 0) = 0 \qquad \forall \lambda \in B_\varepsilon^{\mathbb{R}}(\lambda^*)$$

$$d_v \gamma(\lambda^*, 0) = 0$$

$$\left(d_v \gamma(\lambda^*, 0)[\hat{v}] = -[d_w \Phi(\cdot)]^{-1} d_v \Phi[\hat{v}] \right.$$
$$\left. = -[\quad]^{-1} L_{\lambda^*} \hat{v} = 0 \right)$$

So we have solved for w, now we can substitute $w = \gamma(\lambda, v)$ in the P eq

$$P \, F(\lambda, v + \gamma(\lambda, v)) = 0$$

BIFURCATION EQ

We are in good pos if the P-eq is simpler then the original one.

This is the case for example if

$$\dim \ker L \;, \quad \operatorname{codim} \operatorname{Im} L < \infty$$

Let us see an example when these dim are 1.

Thm (Crandall – Rabinowitz) $F \in C^2(\mathbb{R} \times \mathcal{U}, Y)$,

of $\mathcal{U}$ open in X with $F(\lambda, 0) = 0 \quad \forall \lambda$

Assume that

(1) $V = \ker d_u F(\lambda^*, 0) = \langle u^* \rangle$ (1-dim)

(2) $R = \operatorname{Im} d_u F(\lambda^*, 0)$ closed with codim 1 ($\dim Y/_R = 1$)

(3) $(\partial_{\lambda, u} F)(\lambda^*, 0)[u^*] \notin R$ TRASVERSALITY CONDITION

then $(\lambda^*, 0)$ is a bif. point and $\exists\ \varepsilon_0 > 0$ and a local bif curve parametrized by $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$

$\{(\lambda(\varepsilon), u(\varepsilon)) \in \mathbb{R} \times X\}$ of class C^1

so that $F(\lambda(\varepsilon), u(\varepsilon)) = 0 \quad \forall\ |\varepsilon| \leq \varepsilon_0$

and
$$\begin{cases} \lambda(\varepsilon) = \lambda^* + o(\varepsilon) \\ u(\varepsilon) = \varepsilon u^* + o(\varepsilon^2) \end{cases}$$

Finally $\{(\lambda, u) \in \mathbb{R} \times \mathcal{U} : u \neq 0 \text{ and } F(\lambda, u) = 0\}$

$$\cup\!\!\!\mid$$

$\{(\lambda(\varepsilon), u(\varepsilon)) \in \mathcal{U} : |\varepsilon| \leq \varepsilon_0\}$

proof We apply Lyapunov – Schmidt decomposition

proj complem range $\quad P F(\lambda, v + w) = 0$, $\quad v \in V$ (kernel), $w \in W$ (complement of kernel)

proj. range $\quad Q F(\lambda, v + w) = 0$

We solve the range eq: $w = \gamma(\lambda, v)$ solves Q – eq:

$Q F(\lambda, v + \gamma(\lambda, v)) = 0 \quad \forall \lambda, v$ in a certain neighb.

So we need to solve: $\quad R\, F(\lambda, v + \gamma(\lambda, v)) = 0$

Let us compute P: recall $Y = R + Z$, ← range, ← complement

codim $R = 1 \Rightarrow Z = \langle z_0 \rangle$ 1-dim

$\forall y \in Y$: $\quad y = \alpha z_0 + r \quad , r \in R \quad , \alpha \in \mathbb{R}$

So the operator $P: Y \to Z$, so we need α

$y \longmapsto \alpha z_0$

R closed lin sub $\Rightarrow$ by Hahn-Banach , $\exists\, y^* \in Y^*$, $y^* \neq 0$ ⇝ $y^*(z_0) \neq 0$

with $y^*|_R = 0$

$\rightsquigarrow \quad y^*(y) = y^*(\alpha z_0 + r) = \alpha\, y^*(z_0) \rightsquigarrow \alpha = \dfrac{y^*(y)}{y^*(z_0)}$

So we find: $\quad P y = 0 \iff y^*(y) = 0$

$\rightsquigarrow \quad P F(\lambda, v + \gamma(\lambda,v)) = 0 \iff y^*(F(\lambda, v + \gamma(\lambda,v))) = 0$

We set $\lambda = \lambda^* + \mu$ and since $v \in \ker d_v F(\lambda^*, 0) = \langle u^* \rangle$

we can write $v = t u^*$ and put

$$\beta(\mu, t) = y^*\Big(F(\lambda^* + \mu,\ t u^* + \gamma(\lambda^* + \mu, t u^*)\Big)$$

β is real valued and defined in a neigh of $(0,0)$ in $\mathbb{R} \times \mathbb{R}$

and it is of class C^2 , since F, γ are C^2

Goal: solve $\beta(\mu, t) = 0$ using classical IFT

In particular we want, given t, find $\mu(t)$ solving

$\beta(\mu(t), t) = 0$

Problem $(\beta 1)$ $\quad \beta(\mu, 0) = y^*\big(F(\lambda^* + \mu, \overbrace{\gamma^*(\lambda^* + \mu, 0)}^{"0"})\big) = 0 \quad \forall \mu$

$(\beta 2)$ $\quad \partial_\mu \beta(0,0) = 0$

(CHECK THEM!)

$(\beta 3)$ $\quad \partial_t \beta(0,0) = 0$

So no IFT immediately: we need to "desingularize" $\beta(\mu, t)$

Define

$$h(\mu, t) := \begin{cases} \dfrac{\beta(\mu,t)}{t} & \text{for } t \neq 0 \\ \partial_t \beta(\mu, 0) & \text{for } t = 0 \end{cases}$$

Then for $t \neq 0$ $\quad h(\mu, t) = 0 \iff \beta(\mu, t) = 0$

So we try to apply IFT to h

Note that $h \in C^1$ and $h(0,0) = \partial_t \beta(0,0) = 0$

Moreover:

$$\partial_\mu h(0,0) = \partial_{\mu, t} \beta(0,0)$$

$$\partial_t h(\mu, 0) = \lim_{\varepsilon \to 0} \frac{h(\mu, \varepsilon) - h(\mu, 0)}{\varepsilon} =$$

$$= \lim_{\varepsilon \to 0} \frac{\frac{\beta(\mu, \varepsilon)}{\varepsilon} - \partial_t \beta(\mu, 0)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \Big(\beta(\mu, \varepsilon) - \underbrace{\beta(\mu, 0)}_{=0} - \varepsilon \partial_t \beta(\mu, 0) \Big)$$

$$= \frac{1}{2} (\partial_t^2 \beta)(\mu, 0)$$

So we need to compute $\partial_{t,\mu} \beta(0,0)$ and $\partial_t^2 \beta(0,0)$

We have

$$(\beta 4) \quad \partial_{t,\mu} \beta(0,0) = y^*\left(\partial_{u,\lambda} F(\lambda^*,0)[u^*] \right)$$

$$(\beta 5) \quad \partial_{t,t} \beta(0,0) = y^*\left(\partial_{u,u} F(\lambda^*,0)[u^*,u^*] \right)$$

Let us prove $(\beta 4)$:

$$\partial_t \beta(\mu,t) = y^*\left(d_u F(\lambda^*+\mu, tu^* + \gamma(\lambda^*+\mu, tu^*)) \left[u^* + d_u \gamma(\lambda^*+\mu, tu^*)[u^*] \right] \right)$$

$$\rightsquigarrow \partial_t \beta(\mu,0) = y^*\Big(d_u F(\lambda^*+\mu, \underbrace{\gamma(\lambda^*+\mu,0)}_{=0}) \left[u^* + d_u \gamma(\lambda^*+\mu,0)[u^*] \right] \Big)$$

$$= y^*\left(d_u F(\lambda^*+\mu,0) \left[u^* + d_u \gamma(\lambda^*+\mu,0)[u^*] \right] \right)$$

Now take ∂_μ:

$$\partial_\mu \partial_t \beta(\mu,0) = y^*\left(\partial_{\mu,u} F(\lambda^*+\mu,0) \left[u^* + d_u \gamma(\lambda^*+\mu,0)[u^*] \right] \right)$$

$$+ y^*\left(d_u F(\lambda^*+\mu,0) \left[\partial_{\mu,u} \gamma(\lambda^*+\mu,0)[u^*] \right] \right)$$

$$\rightsquigarrow \partial_\mu \partial_t \beta(0,0) = y^*\Big(\partial_{\mu,u} F(\lambda^*,0) \big[u^* + \underbrace{d_u \gamma(\lambda^*,0)[u^*]}_{=0} \big] \Big)$$

$$\underbrace{+ y^*\left(d_u F(\lambda^*,0) \left[\partial_{\mu,u} \gamma(\lambda^*,0)[u^*] \right] \right)}_{=0}$$

$$= y^*\left(\partial_{\mu,u} F(\lambda^*,0)[u^*] \right)$$

$(\beta 5)$ is proved similarly.

Going back to $h(\mu,t)$. We have that

$$\partial_\mu h(0,0) = y^*\left(\partial_{\mu,u} F(\lambda^*,0)[u^*] \right)$$

By assumption, $\partial_{\mu,u} F(\lambda^*,0)[u^*] \notin R \rightsquigarrow \partial_\mu h(0,0) \neq 0$

We can apply IFT to h and get a map

$$\begin{aligned} (-\varepsilon, \varepsilon) &\longrightarrow \mathbb{R} \\ t &\longmapsto \mu(t) \end{aligned} \quad \text{with} \quad \left| \begin{array}{l} \mu(0) = 0 \\ h(\mu(t), t) = 0 \quad \forall\ |t| \leq \varepsilon \\ t \to \mu(t) \in \mathcal{C}^1 \end{array} \right.$$

$\rightsquigarrow$ so in particular, for $t \neq 0$ we have

$$\beta(\mu(t), t) = 0 \qquad \forall\ |t| \leq \varepsilon$$

So we have solved the P eq!

$$\rightsquigarrow \quad F\Big(\underbrace{\lambda^* + \mu(t)}_{\lambda(t)},\ \underbrace{tu^* + \gamma(\lambda^* + \mu(t),\ tu^*)}_{u(t)}\Big) = 0 \qquad \forall\ |t| \leq \varepsilon$$

and it is not trivial since $\gamma(\lambda^* + \mu(t), tu^*) = O(t^2)$

$\blacksquare$

Application Non linear Sturm-Liouville problem

$$(*) \begin{cases} -u'' + \lambda u + f(u) = 0 \\ u(0) = u(1) = 0 \end{cases}, \quad f \in C^2, \quad \begin{matrix} f(0) = 0 \\ f'(0) = 0 \end{matrix}$$

Solutions are zeroes of

$$F(\lambda, u) = -u'' + \lambda u + f(u), \quad F : \mathbb{R} \times X \to Y$$

$$X = \{ u \in C^2([0,1]) : \quad u(0) = u(1) = 0 \}$$

$$Y = \{ u \in C^0([0,1]) \}$$

Clearly $\quad F(\lambda, 0) = 0 \quad \forall \lambda$

Bifurcation values: $\quad d_u F(\lambda^*, 0)[h] = -h'' + \lambda h + f'(0) \cdot h$

$$= -h'' + \lambda h$$

not invertible

$d_u F(\lambda^*, 0)$ invertible $\iff \forall g \in C^0, \exists ! \, h \in C^2$ sol of

$$\begin{cases} -h'' + \lambda h = g \\ h(0) = h(1) = 0 \end{cases}$$

$\iff$ hom problem has only trivial sol

$$\begin{cases} -h'' + \lambda h = 0 \\ h(0) = h(1) = 0 \end{cases}$$

$\iff \lambda \notin \{ -n^2 \pi^2, \; n \in \mathbb{N} \}$

Moreover, if $\lambda = -n^2\pi^2$, $\exists$ sol of $\begin{cases} -u'' + \lambda u = f \\ u(0) = u(1) = 0 \end{cases}$

$\Longleftrightarrow$ $f \perp$ sol of homog problem with $\lambda = -n^2\pi^2$

$\Longleftrightarrow$ $\langle f, \underbrace{\sin(n\pi x)}_{u_*} \rangle = 0$

Equivalently $f \in \operatorname{Im} d_u F(\lambda_n^*, 0)$ $\Longleftrightarrow$ $f \perp \sin(n\pi x)$
$\lambda_n^* = -n^2\pi^2$

Can we apply Crandall-Rabinowitz to find solutions in case $\lambda = -n^2\pi^2$?

Put $V = \ker d_u F(\lambda_n^*, 0) = \langle \sin(n\pi x) \rangle$

$R = \operatorname{Im} d_u F(\lambda_n^*, 0) = \{ f \in Y : \int f(x) \sin(n\pi x) = 0 \}$

$\rightsquigarrow$ $\dim V = \operatorname{codim} R = 1$, R closed

$R = \varphi^{-1}(0)$ with
$\varphi : Y \to \mathbb{R}$ lin. funct
$\varphi(f) = \int f(x) \sin(n\pi x)$

It remains to check the transversality condition:

$$(\partial_{\lambda,u} F)(\lambda_n^*, 0)[u^*] \notin R$$

As $(\partial_\lambda d_u F)(\lambda_n^*, 0)[\hat{h}] = \partial_\lambda (-\hat{h}'' + \lambda \hat{h}) = \hat{h}$,
we need $(\partial_\lambda d_u F)(\lambda_n^*, 0)[u^*] = u^* \notin R$

But $u^* \in R \Longleftrightarrow \int u^*(x)^2 = 0$ FALSE! So Transversality condition ✓!

CR $\Rightarrow$ $\forall n$, $\exists$ a continuous family of nontrivial sol of $(*)$ with

$$\begin{cases} u(\varepsilon) = \varepsilon \sin(n\pi x) + o(\varepsilon^2) \\ \lambda(\varepsilon) = -n^2\pi^2 + o(\varepsilon) \end{cases}$$